

$\bar{\partial}$ -EQUATION ON (p, q) -FORMS ON CONIC NEIGHBOURHOODS OF 1-CONVEX MANIFOLDS

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ABSTRACT. Let X be a 1-convex manifold with the exceptional set S , which is also a manifold, Z a complex manifold, $Z \rightarrow X$ a holomorphic submersion, $a : X \rightarrow Z$ a holomorphic section and $S \subset U \Subset X$ an open relatively compact 1-convex set. We construct a metric on a vector bundle $E \rightarrow Z$ restricted to a neighbourhood V of $a(U)$, conic along $a(S)$ with at most polynomial poles at $a(S)$ and positive Nakano curvature tensor in bidegree (p, q) .

1. INTRODUCTION AND THE MAIN THEOREM

Let $\pi : Z \rightarrow X$ be a submersion from a complex manifold Z to a 1-convex manifold X with the exceptional set S , which is also a manifold. The motivation for this work was to construct solutions of the $\bar{\partial}$ -equation with at most polynomial poles at $\pi^{-1}(S)$ in a particular geometric situation (see Fig. 1), namely on a conic neighbourhood V of sections of $\pi : Z \rightarrow X$. The results of this paper may provide one step in the proof of such a claim. It turned out that it is possible to construct solutions of the $\bar{\partial}$ -equation such that their L^2 -norms on $V_\delta := \{z \in V, d(z, \pi^{-1}(S)) > \delta\}$ with respect to some ambient Hermitian metric h_Z on Z and h_E on E grow at most polynomially as $\delta \rightarrow 0$. To this end weights that give rise to positive Nakano curvature and have polynomial behaviour on $\pi^{-1}(S)$ are constructed. Sup-norm estimates are given in the case $q = 1$. The main theorem of the present paper is the following:

Theorem 1.1 (Nakano positive curvature tensor in bidegree (p, q)). *Let Z be a n -dimensional complex manifold, X a 1-convex manifold, $S \subset X$ its exceptional set, which is also a manifold, $\pi : Z \rightarrow X$ a holomorphic submersion with r_0 -dimensional fibres, $\sigma : E \rightarrow Z$ a holomorphic vector bundle and $a : X \rightarrow Z$ a holomorphic section. Let $\varphi : X \rightarrow [0, \infty)$ be a plurisubharmonic exhaustion function, strictly plurisubharmonic on $X \setminus S$ and $\varphi^{-1}(0) = S$. Let $U = \varphi^{-1}([0, c])$ for some $c > 0$ be a given holomorphically convex set and let $s \in \{1, \dots, n\}$. Then there exist an open neighbourhood V of $a(U \setminus S)$ conic along $a(S)$, a Nakano positive Hermitian metric h on $E|_V$ with at most polynomial poles on $\pi^{-1}(S)$ and such that the Chern curvature tensor $i\Theta(E \otimes \Lambda^s T Z)$ restricted to V is Nakano positive and has at most polynomial poles and zeroes on $\pi^{-1}(S)$.*

For standard techniques for solving the $\bar{\partial}$ -equation we refer to Demailly's book Complex analytic and algebraic geometry [Dem]. Our main tool is

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Theorem 1.2 (Theorem VIII-4.5, [Dem]). *If (W, ω) is complete and $A_{E, \omega} > 0$ in bidegree (p, q) , then for any $\bar{\partial}$ -closed form $u \in L^2_{p, q}(W, E)$ with*

$$\int_W \langle A_{E, \omega}^{-1} u, u \rangle dV < \infty$$

there exists $v \in L^2_{p, q-1}(W, E)$ such that $\bar{\partial}v = u$ and

$$\|v\|^2 \leq \int_W \langle A_{E, \omega}^{-1} u, u \rangle dV.$$

It seems difficult to handle the commutator $A_{E, \omega}$ (see Sect. VII-7, [Dem] for computations) in the case of (p, q) -forms because it has terms of mixed signs for $p < n$. Therefore we view the (p, q) -form u as a (n, q) -form u_1 with coefficients in $E_1 = E \otimes \Lambda^{n-p} T^*Z$ by invoking the isomorphism $\Lambda^p T^*Z \simeq \Lambda^{n-p} T^*Z \otimes \Lambda^n T^*Z$. If u is closed so is u_1 . In addition, in bidegree (n, q) , there is an analog of Theorem 1.2 for a noncomplete Kähler metric provided that the manifold possesses a complete one (Theorem VIII-6.1, [Dem]). Moreover, the positivity of $A_{E_1, \omega}$ follows from the positivity of the Chern curvature tensor $i\Theta(E_1) = i\Theta(E) + i\Theta(\Lambda^{n-p} T^*Z)$.

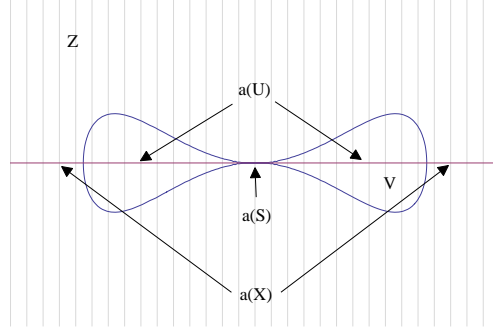


FIGURE 1. Conic neighbourhoods of $a(U \setminus S)$ in Z

To use both theorems we first need a Nakano positive Hermitian metric on $E|_V$ and a Kähler metric on V , both with polynomial zeroes or poles on $\pi^{-1}(S)$. The Kähler metric $\omega = i\partial\bar{\partial}\Phi$ constructed in Sect. 2 and the Nakano positive Hermitian metric given by Theorem 1.1 in [Pre2] have the desired properties. The space (V, ω) is not complete but if we take a smaller neighbourhood conic along $a(N(g))$, $N(g) := g^{-1}(0)$, for some holomorphic function $g : X \rightarrow \mathbb{C}$ with $g(S) = 0$, it contains a conic Stein neighbourhood V' which is complete Kähler (Fig. 2). With the above notation we have

Corollary 1.3 ($\bar{\partial}$ -equation in bidegree (p, q)). *Let $g : X \rightarrow \mathbb{C}$ be holomorphic with $N(g) = g^{-1}(0) \supset S$. Let V' be an open Stein neighbourhood of $a(U \setminus N(g))$, conic along $a(N(g))$, let u be a closed (p, q) -form on V' , u_1 the corresponding (n, q) -form with coefficients in $E_1 = E \otimes \Lambda^{n-p} T^*Z$ and let h be the metric on E from Theorem 1.1 with $s = n - p$ and h_1 the induced metric on E_1 . Denote by $A = A_{E \otimes \Lambda^{n-p} T^*Z, \omega}$ the commutator and assume that*

$$\int_{V'} \langle A^{-1} u_1, u_1 \rangle_{h_1} dV_\omega < \infty$$

Then there exist an $(n, q - 1)$ -form v_1 with

$$\|v_1\|^2 = \int_{V'} \langle v_1, v_1 \rangle_{h_1} dV_\omega \leq \int_{V'} \langle A^{-1}u_1, u_1 \rangle_{h_1} dV_\omega.$$

Corollary 1.4. *If u is smooth and v_1 is the minimal norm solution then v_1 and the associated $(p, q - 1)$ -form v are smooth. The L^2 -norms of v_1 on the sets $V'_\delta := \{z \in V', d(z, \pi^{-1}(S)) > \delta\}$ with respect to h_Z and h_E grow at most polynomially with respect to δ as $\delta \rightarrow 0$ and the same holds for the L^2 -norms of the corresponding $(p, q - 1)$ -form v .*

If $q = 1$ we can use Lemma 4.5 in [Pre2] which is an adaptation of Lemma 3.2 in [?] to get sup-norm estimates from the Bochner-Martinelli-Koppelman formula if we take a slightly smaller Stein neighbourhood $V'' \subset V'$, conic along $a(N(g))$ (Fig. 2)

Corollary 1.5. *If $q = 0$ and the initial form u is smooth on a neighbourhood of $a(\bar{U} \setminus S)$, the form v has at most polynomial poles on $\pi^{-1}(S)$.*

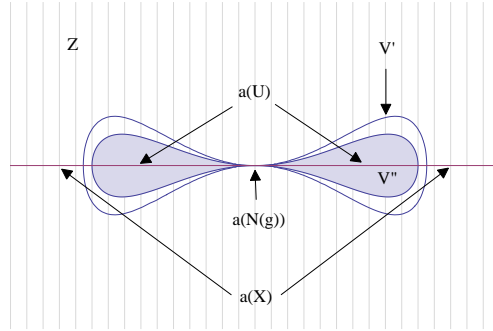


FIGURE 2. Conic neighbourhoods of $a(U \setminus N(g))$ in Z

Notation. The notation from Theorem 1.1 is fixed throughout the paper. Let h_Z be a Hermitian metric defined on the manifold Z and let $\sigma : E \rightarrow Z$ be a holomorphic vector bundle of rank r equipped with a Hermitian metric h_E . The local coordinate system in a neighbourhood $V_{z_0} \subset Z$ of a point $z_0 \in a(U)$ is (z, w) , where z denotes the horizontal and w the vertical (or fibre) direction and $z_0 = (0, 0)$. More precisely, every point in $a(U)$ has $w = 0$ and points in the same fibre have the same first coordinate. If the point z_0 is in $a(S)$ we write the z -coordinate as $z = (z_1, z_2)$, where $a(S) \cap V_{z_0} = \{z_2 = 0, w = 0\} \cap V_{z_0}$. The manifold Z is n -dimensional and the dimension of the fibres Z_{z_0} is constant, $r_0 = \dim Z_{z_0}$. The notation ζ_1, \dots, ζ_n is sometimes used for local coordinates in Z .

2. CONSTRUCTION OF THE KÄHLER METRIC ω

The Kähler metric ω will be obtained from the Kähler potential $\Phi = \varphi_0 + \varphi_1$ and is similar to the one constructed in subsection 2.1 in [Pre2]. The only difference is that we choose a specific plurisubharmonic function φ_0 instead of the given φ in order to be able to study the curvature properties of ω . The construction is explained below.

Since the Remmert reduction $p : X \rightarrow \hat{X}$ of X is a Stein space with finitely many isolated singular points it has a proper holomorphic embedding $\hat{f} : \hat{X} \rightarrow \mathbb{C}^M$ for some large M .

Consequently, the holomorphic functions $\hat{f}_1 \circ p, \dots, \hat{f}_M \circ p$ generate the cotangent space $T^*X \setminus S$ and the function $\hat{\varphi}_0 := \sum |\hat{f}_i \circ p|^2$ is a plurisubharmonic exhaustion function of X , strictly plurisubharmonic on $X \setminus S$. The functions $f_i := \hat{f}_i \circ p \circ \pi$ are defined on Z and they generate the horizontal cotangent space on $a(X \setminus S)$. Define $\varphi_0 := \hat{\varphi}_0 \circ \pi = \sum |f_i|^2$ and denote by k_0 the maximal order of degeneracy of $i\partial\bar{\partial}\varphi_0$ at $\pi^{-1}(S)$.

For given $l > 2$ the construction in Subsection 2.1 in [Pre2] with the function φ replaced by φ_0 yields almost holomorphic functions f_{M+1}, \dots, f_N , holomorphic to a degree l with zeroes of order at least k on $\pi^{-1}(S)$. To be precise, for every sufficiently large k by Theorem A there exist sections $F_{M+1}, \dots, F_N \in \Gamma(U', \mathcal{J}(a(S))^k(\mathcal{J}(a(U'))/\mathcal{J}^{l+1}(a(U'))))$, $U \Subset U'$ which locally generate the sheaf on \overline{U} . We further assume that $k > k_0$. The functions f_{M+1}, \dots, f_N , are obtained by patching together particular local lifts of these sections using the partition of unity $\{\chi_j, U_j\}$ which we (can) choose to depend on the horizontal variables only and so f_{M+1}, \dots, f_N , are holomorphic in vertical directions. In local coordinates (z, w) near $a(S)$ we thus have

$$f_i(z, w) = \sum_{|\alpha|=k, 0 < |\beta| \leq l} c_{\alpha\beta} z_2^\alpha w^\beta + \sum_{i,j,l} \chi_j(z) f_{ijl}(z, w)$$

with $f_{ijl} \in \mathcal{O}(\|z_2\|^k \|w\|^{l+1})$ holomorphic on open sets U_j . The functions f_i satisfy the estimate $\bar{\partial} f_i(z, w) \approx \|z_2\|^k \|w\|^{l+1}$. Moreover, we can express $z_2^\alpha w_j = \sum g_{\alpha ij}(z) f_i(z, w) + \mathcal{O}(\|z_2\|^k \|w\|^{l+1})$ with $g_{\alpha ij}$ holomorphic and from this we infer that $\partial_{w_j} f_i$, $1 \leq j \leq r_0, M+1 \leq i \leq N$ generate the vertical cotangent bundle on a neighbourhood V_T of $a(\overline{U})$ in Z except on $\pi^{-1}(S)$. Consequently, the matrix corresponding to $\partial_w \bar{\partial}_w \sum |f_i|^2$ is of the form $\|z_2\|^{2k} G$, with G invertible. Define

$$\Phi := \sum f_i \bar{f}_i \text{ and } \omega := i\partial\bar{\partial}\Phi.$$

We claim that the function Φ is a Kähler potential on a conic neighbourhood of $a(U \setminus S)$ and ω a Kähler metric.

Write local coordinates as $(\zeta_1, \dots, \zeta_n) = (z, w)$ and represent the Levi form

$$i\partial\bar{\partial}\Phi = i \sum h_{jk} d\zeta_j \wedge d\bar{\zeta}_k$$

by a matrix $H = \{h_{jk}\}$. In local coordinates (z, w) the nonnegative part of ω , $\omega_+ = i \sum \partial f_i \wedge \bar{\partial} \bar{f}_i$, represented in the matrix form as H_+ , can be estimated from below by

$$(2.1) \quad H_+(z, w) \gtrsim \begin{bmatrix} \|z_2\|^{2k_0} + \|w\|^2 \|z_2\|^{2k-2} & \|w\| \|z_2\|^{2k-1} \\ \|w\| \|z_2\|^{2k-1} & \|z_2\|^{2k} \end{bmatrix},$$

where we have estimated the decay of φ_0 by $\|z_2\|^{2k_0}$ from below. The possibly negative part $\omega_- = i \sum \partial \bar{\partial} f_i \bar{f}_i + f_i \partial \bar{\partial} \bar{f}_i + \partial \bar{f}_i \wedge \bar{\partial} f_i$ degenerates at least as $\|w\|^l \|z_2\|^{k-1}$. It is clear that for $\|z_2\| > \delta$ and small $\|w\|$ or $\|z_2\| \leq \delta$ and $\|w\| \leq \|z_2\|^2$ the matrix H is strictly positive definite and thus ω is a Kähler metric on a neighbourhood of $a(U \setminus S)$, conic along $a(S)$. Let (z, w) be local coordinates near $a(S)$ and define $H_0(z) := H(z, 0) = H_+(z, 0)$, $H_1 = H - H_0$. It follows that $H_1 = \mathcal{O}(\|w\| \|z_2\|^{2k-1})$ and that H_0 decreases polynomially (in some directions) as we approach $\pi^{-1}(S)$ and its degeneracy is bounded from below by $\|z_2\|^{2k}$,

$$H_0(z) \gtrsim \begin{bmatrix} \|z_2\|^{2k_0} & 0 \\ 0 & \|z_2\|^{2k} \end{bmatrix}.$$

Notice that H_0 is strictly positive on a neighbourhood of $a(\overline{U})$ except on $\pi^{-1}(S)$ (and therefore invertible) and $\|H_0^{-1}\|$ degenerates in the worst case as $\|z_2\|^{-\kappa}$ for some $\kappa \geq 0$. Because S is compact, there exists one κ for all points in $a(S)$. Write $H = H_0(I + H_0^{-1}H_1)$, $H^{-1} = (I + H_0^{-1}H_1)^{-1}H_0^{-1}$, then

$$\|H_0^{-1}H_1\| \approx \|z_2\|^{2k-1-\kappa}\|w\|$$

in the worst case and this term is small, $\|H_0^{-1}H_1\| < \|z_2\|^{3\kappa+k_1}$ on conic neighbourhoods of the form $\|w\| \leq \|z_2\|^{4\kappa+k_1}$, and so

$$(2.2) \quad H^{-1} = H_0^{-1} + H_0^{-1}H_1 \sum_{n=0}^{\infty} (H_0^{-1}H_1)^n H_0^{-1} = H_0^{-1} + N, \quad \|N\| \leq \|z_2\|^{2\kappa+k_1}.$$

Inside this cone the degeneracy of the inverse H^{-1} is governed by H_0^{-1} .

3. THEOREMS ON CURVATURES

3.1. Basic theorems on curvatures. Before proceeding to the proof we recall some formulae from Demailly's Complex analytic and algebraic geometry [Dem].

Let (X, ω) be a Kähler manifold, $E \rightarrow X$ a rank r vector bundle equipped with a Hermitian metric h . The matrix H that corresponds to h in local coordinates is given by $\langle u, v \rangle_h = \sum h_{\lambda\mu} u_{\lambda} \overline{v_{\mu}} = u^T H \overline{v}$. Let $i\Theta(E)$ be the Chern curvature form of the metric and Λ the adjoint of the operator $u \rightarrow u \wedge \omega$, defined on (p, q) -forms. Denote by $L_{p,q}^2(X, E)$ the space of (p, q) -forms with bounded L^2 -norms with respect to the h and let $A_{E,\omega} = [i\Theta(E), \Lambda]$ be the commutator.

In bidegree (n, q) the positivity of $A_{E,\omega}$ is equivalent to Nakano positivity of E . Let e_1, \dots, e_r be a local frame of E . If the metric is locally represented by a matrix H then

$$(3.1) \quad i\Theta(E) = i\overline{\partial}(\overline{H}^{-1}\partial\overline{H}) = i \sum c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e_{\lambda}^* \otimes e_{\mu},$$

If e_1, \dots, e_r is an orthonormal frame then the Hermitian form θ_E defined on $TX \otimes E$, which is associated to $i\Theta(E)$, takes the form

$$(3.2) \quad \theta_E = \sum c_{jk\lambda\mu} (dz_j \otimes e_{\lambda}^*) \otimes \overline{(dz_k \otimes e_{\mu}^*)}.$$

The curvature tensor (3.1) is *Griffiths positive* if the form (3.2) is positive on decomposable tensors $\tau = \xi \otimes v$, $\xi \in TX$, $v \in E$, $\theta_E(\tau, \tau) = \sum c_{jk\lambda\mu} \xi_j \overline{\xi_k} v_{\lambda} \overline{v_{\mu}}$ and *Nakano positive* if it is positive on $\tau = \sum \tau_{j\lambda} (\partial/\partial z_j) \otimes e_{\lambda}$, $\theta_E(\tau, \tau) = \sum c_{jk\lambda\mu} \tau_{j\lambda} \overline{\tau_{k\mu}}$. In a nonorthonormal frame we have $\theta_E(\tau, \tau) = \sum c_{jk\lambda\mu} \tau_{j\lambda} \overline{\tau_{k\mu}} h_{\mu\nu}$. Proposition VII-9.1, [Dem] states that

$$(3.3) \quad \text{if } \theta(E) >_{\text{Grif}} 0 \text{ then } r \text{Tr}_E(\theta(E)) \otimes h - \theta(E) >_{\text{Nak}} 0.$$

The metric h on E induces the metric h^s on $\Lambda^s E$. Let L be an s -tuple of (not necessarily ordered) indices $L = (\lambda_1, \dots, \lambda_s)$ and denote $e_L := e_{\lambda_1} \wedge \dots \wedge e_{\lambda_s}$. If σ is a permutation then $e_{\sigma(L)} = \text{sign}(\sigma) e_L$. Let $L, M \in \{(\lambda_1, \dots, \lambda_s), 1 \leq \lambda_1 < \dots < \lambda_s \leq r\} =: \mathcal{L}$, $L = (\lambda_1, \dots, \lambda_s)$, $M = (\mu_1, \dots, \mu_s)$. The coefficient $H_{LM} = \langle e_L, e_M \rangle_{h^s}$ in the matrix H^s representing the induced metric h^s is

$$H_{LM}^s = \det H_{(\lambda_1, \dots, \lambda_s), (\mu_1, \dots, \mu_s)},$$

where $H_{(\lambda_1, \dots, \lambda_s), (\mu_1, \dots, \mu_s)}$ is a submatrix of H generated by rows $\lambda_1, \dots, \lambda_s$ and columns μ_1, \dots, μ_s of the matrix H . If e_1, \dots, e_r are orthonormal at z so are their wedge products $\{e_L, L \in \mathcal{L}\}$.

The induced Chern curvature tensor on $\Lambda^s E$, $i\Theta(\Lambda^s(E)) = \sum_{j,k} i\Theta(E)_{jk} dz_j \wedge d\bar{z}_k$ is defined by formula V-(4.5'), [Dem],

$$(3.4) \quad i\Theta(\Lambda^s(E))_{jk}(e_L) = i \sum_{1 \leq l \leq s} e_{\lambda_l} \wedge \dots \wedge \Theta(E)_{jk} e_{\lambda_l} \wedge \dots \wedge e_{\lambda_s}.$$

It is known that $E \geq_{\text{Nak}} 0$ implies $\Lambda^s E \geq_{\text{Nak}} 0$. The following lemma gives an explicit formula for the curvature $i\Theta(\Lambda^s(E))$ in terms of the curvature $i\Theta(E)$ and shows that if the associate Hermitian form $\theta(E)$ has at most polynomial poles on $\pi^{-1}(S)$, so does $\theta(\Lambda^s(E))$.

Lemma 3.1. *If $i\Theta(E)$ is Nakano nonpositive (nonnegative) then $i\Theta(\Lambda^s(E))$, $1 \leq s \leq r$, is also Nakano nonpositive (nonnegative).*

Proof. By formula (3.4) we have

$$i\Theta(\Lambda^s(E))(e_L)_{jk} = i \sum_{l,\mu} (-1)^{l-1} c_{jk\lambda_l\mu} e_\mu \wedge e_{L'_l},$$

where L'_l is obtained from L by removing the l -th index. Let $L(\lambda, \mu)$ denote the (not ordered) multiindex obtained by replacing the index λ in the multiindex L by μ . We define that $e_{L(\lambda,\mu)} = 0$ if and only if $\lambda \notin L$ or $\mu \in L \setminus \{\lambda\}$. If $\lambda_l = \lambda$ then $e_{\mu L'_l} = (-1)^{l-1} e_{L(\lambda,\mu)}$ and

$$\begin{aligned} i\Theta(\Lambda^s(E)) &= i \sum_{j,k,L,M} c_{jkLM}^s dz_j \wedge d\bar{z}_k \otimes e_L^* \otimes e_M \\ &= i \sum_{j,k,L} \sum_{\lambda \in L, \mu} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_L^* \otimes e_{L(\lambda,\mu)} \\ &= i \sum_{\substack{j,k \\ |L'|=s-1}} \sum_{\lambda, \mu \notin L'} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_{\lambda L'}^* \otimes e_{\mu L'}. \end{aligned}$$

Here the bijection between the sets $\{L : |L| = s\}$ and $\{\lambda L' : |L'| = s-1, \lambda \notin L'\}$ is used, where we compare multiindices as sets. Let $\tau = \sum_{j,L} \tau_{j,L} (\partial/\partial z_j) \otimes e_L$ with additional properties $\tau_{j,\sigma(L)} := \text{sign}(\sigma) \tau_{j,L}$ for any permutation σ and $\tau_{j,L} = 0$ if there are at least two equal indices in L . The bundle $\Lambda^s(E)$ is Nakano positive if the bilinear form

$$\begin{aligned} \theta_{\Lambda^s(E)}(\tau, \tau) &= \sum_{j,k,L,M} c_{jkLM}^s \tau_{j,L} \bar{\tau}_{k,S} \langle e_M, e_S \rangle_{h^s} \\ &= \sum_{\substack{j,k, |S|=s \\ |L'|=s-1}} \sum_{\lambda, \mu \notin L'} c_{jk\lambda\mu} \tau_{j,\lambda L'} \bar{\tau}_{k,\mu L'} \langle e_{\mu L'}, e_S \rangle_{h^s} \end{aligned}$$

is positive. Assume that the local frame e_1, \dots, e_r is orthonormal. Then $\{e_L, L \in \mathcal{L}\}$ are orthonormal and

$$(3.5) \quad \theta_{\Lambda^s(E)}(\tau, \tau) = \sum_{j,k,L'} \sum_{\lambda,\mu} c_{jk\lambda\mu} \tau_{j,\lambda L'} \bar{\tau}_{k,\mu L'} = \sum_{L'} \theta_E(\tau_{L'}, \tau_{L'}),$$

where the form $\tau_{L'}$ is defined by $\tau_{L'} = \sum_{j,\lambda} \tau_{j,\lambda L'} (\partial/\partial z_j) \otimes e_\lambda$ for any multiindex L' of length $s-1$. Hence if θ_E is Nakano nonpositive (nonnegative), so is $\theta_{\Lambda^s E}$. \square

3.2. Almost nonpositivity of ω . In this section we study properties of the form ω constructed in Sect. 2. Let $V \subset Z$ be a neighbourhood of $a(U \setminus S)$, conic along $a(S)$. The metric ω on a vector bundle $E \rightarrow V$ is *almost Nakano nonpositive* if the curvature tensor $i\Theta(E)$ has a decomposition $i\Theta(E) = i\Theta_0(E) + i\Theta_1(E)$, where $i\Theta_0(E)$ is nonpositive and $i\Theta_1(E)$ is locally of the form $i\Theta_1(E)(z, w) = \mathcal{O}(\|w\|^l \|z_2\|^k)$ near points in $a(S)$ and $i\Theta_1(E)(z, w) = \mathcal{O}(\|w\|^l)$ near points in $a(U \setminus S)$, for some $l \in \mathbb{N}$, $k \in \mathbb{Z}$.

The main theorem in this subsection is

Theorem 3.2 (Almost Nakano nonpositive Kähler metric). *Let Z, X, S, a, U and ω be as in Theorem 1.1. There exist a neighbourhood V of $a(\overline{U} \setminus S)$ conic along $a(S)$ such that the metric ω on $TZ|_V$ is almost Nakano nonpositive.*

Corollary 3.3. *Let h_ω be the metric on $TZ|_V$ induced by ω . Then $h_\omega e^\Phi$ is Nakano negative on a smaller neighbourhood - which we again denote by V - of $a(U \setminus S)$, conic along $a(S)$.*

Proof of Theorem 3.2. Write $f = (f_1, \dots, f_N)^T$ and $\sum |f_i|^2 = f^T \bar{f}$ and let H denote the matrix corresponding to the metric. If D denotes the holomorphic and \bar{D} the antiholomorphic derivative with respect to (z, w) ,

$$Df = \begin{bmatrix} f_{1,z_1} & f_{1,z_2} & f_{1,z_3} & \cdots & f_{1,w_{r_0}} \\ f_{2,z_1} & f_{2,z_2} & f_{2,z_3} & \cdots & f_{2,w_{r_0}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{N,z_1} & f_{N,z_2} & f_{N,z_3} & \cdots & f_{N,w_{r_0}} \end{bmatrix}$$

then

$$H = D\bar{D}(f^T \bar{f}) = (Df)^T \bar{D}\bar{f} + D\bar{f}^T \bar{D}f + f^T L\bar{f} + Lf^T \bar{f}$$

The Levi form Lf of the vector is calculated as $Lf = (Lf_1, \dots, Lf_N)$. With the notation defined prior to (2.1) we have $H_+ = (Df)^T \bar{D}\bar{f}$ and $H_- = H - H_+ = D\bar{f}^T \bar{D}f + f^T L\bar{f} + Lf^T \bar{f}$. Since f_i are holomorphic or almost holomorphic to the degree l , we have $\bar{\partial}f = \mathcal{O}(\|w\|^{l+1})$, terms $\bar{\partial}Df, Lf, H_-$ are of the form $\mathcal{O}(\|w\|^l)$ and because of the holomorphicity of f_i in w -directions the terms $\partial H_-, \bar{\partial}H_-, \partial\bar{\partial}Df, \partial\bar{\partial}\bar{H}_-$ are of the form $\mathcal{O}(\|w\|^{l-1})$. The term H_+ gives

$$\begin{aligned} \partial\bar{\partial}\bar{H}_+ &= \partial\bar{\partial}((Df)^* Df) = -\bar{\partial}\partial((Df)^* Df) = -\bar{\partial}(\partial(Df)^* Df + (Df)^*(\partial Df)) \\ &= -(\partial Df)^* \wedge (\partial Df) + \partial(Df)^* \wedge \bar{\partial}Df + (\partial\bar{\partial}(Df)^*) Df + (Df)^* \partial\bar{\partial}Df \end{aligned}$$

and so

$$(3.6) \quad \partial\bar{\partial}\bar{H} = -(\partial Df)^* \wedge (\partial Df) + \mathcal{O}(\|w\|^{l-1}).$$

By the above estimates we conclude that

$$(3.7) \quad \partial\bar{H} = \partial(\bar{H}_+ + \bar{H}_-) = (Df)^*(\partial Df) + \mathcal{O}(\|w\|^{l-1})$$

and similarly

$$(3.8) \quad \bar{\partial}\bar{H} = (\partial Df)^* Df + \mathcal{O}(\|w\|^{l-1}).$$

Since the metric is Kähler, we may assume that the coordinates ζ_1, \dots, ζ_n near the point $z_0 \in a(U \setminus S)$ are such that $H(z_0) = I$ to the second order. The curvature form equals

$$\begin{aligned} i\Theta(TZ) &= i\bar{\partial}(\bar{H}^{-1}\partial\bar{H}) \\ &= -i\bar{H}^{-1}\bar{\partial}\bar{H}\bar{H}^{-1} \wedge \partial\bar{H} + i\bar{H}^{-1}\bar{\partial}\partial\bar{H} \\ &= -i\bar{\partial}\bar{\partial}\bar{H} = i(\partial Df)^* \wedge (\partial Df) + \mathcal{O}(\|w\|^{l-1}) \end{aligned}$$

by the above assumptions. We claim that $i(\partial Df)^* \wedge (\partial Df)$ is Nakano nonpositive. Denote the dual tangent vectors by $e_\lambda := \partial/\partial\zeta_\lambda$ and let

$$i\Theta_0 = i(\partial Df)^* \wedge (\partial Df) = i \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} d\zeta_j \wedge d\bar{\zeta}_k \otimes e_\lambda^* \otimes e_\mu.$$

Then we have

$$c_{jk\lambda\mu} = - \sum_{i=1}^N \frac{\partial^2 f_i}{\partial\zeta_j \partial\zeta_\lambda} \overline{\frac{\partial^2 f_i}{\partial\zeta_k \partial\zeta_\mu}}$$

and so

$$\theta_0(\tau, \tau) = - \sum_i \left| \sum_{j\lambda} \frac{\partial^2 f_i}{\partial\zeta_j \partial\zeta_\lambda} \tau_{j\lambda} \right|^2 \leq 0.$$

Remark 3.4. If the functions f_i were holomorphic, then the Nakano nonpositivity of $i\Theta(VT)$ on V could be inferred from the fact that the metric on $TZ|_V$ is the metric induced on the subbundle $F \leq V \times \mathbb{C}^N$ by the standard metric on \mathbb{C}^N via the holomorphic vector bundle isomorphism $TZ_V \rightarrow F$, $TZ \ni v \mapsto (df_1(v), \dots, df_N(v)) \in \mathbb{C}^N$ and it is known that the Nakano curvature decreases in subbundles by VII-(6.10), [Dem]. Because f_i are almost holomorphic, we get an ‘error’ term, denoted by $i\Theta_1$ in the sequel and we will show that it decreases arbitrarily fast on conic neighbourhoods. On the section $a(U \setminus S)$ the ‘holomorphic’ part of the curvature tensor is exactly $i\Theta_0$.

The curvature form $i\Theta(TZ)$ is then almost Nakano nonpositive on an open neighbourhood of $a(U \setminus S)$. If we want to prove that the neighbourhood is conic we have to show that the part of the curvature which contains w -variables decreases polynomially sufficiently fast in some conic neighbourhood. Let (z, w) be local coordinates at $(z, 0) \in a(S)$. We can not assume that $H = I$ to the second order at $(z, 0)$ because $H(z, 0)$ is degenerate on $a(S)$.

We have to split the curvature tensor into the part depending only on z -variables - we have just proved that it is nonpositive - and the rest, which we want to be small on conic neighbourhoods. By estimates (2.2), (3.6), (3.7), (3.8) we have for $\|w\| \leq \|z_2\|^{4\kappa+k_1}$

$$\begin{aligned} \bar{H}^{-1}\bar{\partial}\partial\bar{H} &= \bar{H}^{-1}((\partial Df)^* \wedge (\partial Df) + \mathcal{O}(\|w\|^{l-1})) \\ &= \bar{H}_0^{-1}(\partial Df)^* \wedge (\partial Df) + \bar{N}(\partial Df)^* \wedge (\partial Df) + \mathcal{O}(\|w\|^{l-1}\|z_2\|^{-\kappa}) \\ &= \bar{H}_0^{-1}(\partial Df)^* \wedge (\partial Df) + \mathcal{O}(\|z_2\|^{2\kappa+k_1}) + \mathcal{O}(\|w\|^{l-1}\|z_2\|^{-\kappa}), \end{aligned}$$

$$\begin{aligned}
\overline{H}^{-1}\overline{\partial H H}^{-1} \wedge \partial \overline{H} &= (\overline{H}_0^{-1} + \overline{N})((\partial Df)^* Df + \mathcal{O}(\|w\|^{l-1}))(\overline{H}_0^{-1} + \overline{N}) \\
&\quad \wedge ((Df)^*(\partial Df) + \mathcal{O}(\|w\|^{l-1})) \\
&= \overline{H}_0^{-1}(\partial Df)^* Df \overline{H}_0^{-1} \wedge (Df)^*(\partial Df) + \overline{H}_0^{-1}(\partial Df)^* Df \overline{N} \wedge (Df)^*(\partial Df) \\
&\quad + \overline{N}(\partial Df)^* Df \overline{H}_0^{-1} \wedge (Df)^*(\partial Df) + \overline{N}(\partial Df)^* Df \overline{N} \wedge (Df)^*(\partial Df) \\
&\quad + \mathcal{O}(\|z_2\|^{-2\kappa})\mathcal{O}(\|w\|^{l-1}) \\
&= \overline{H}_0^{-1}(\partial Df)^* Df \overline{H}_0^{-1} \wedge (Df)^*(\partial Df) + \mathcal{O}(\|z_2\|^{2\kappa+k_1})\mathcal{O}(\|z_2\|^{-\kappa}) \\
&\quad + \mathcal{O}(\|z_2\|^{2(2\kappa+k_1)}) + \mathcal{O}(\|z_2\|^{-2\kappa})\mathcal{O}(\|w\|^{l-1}).
\end{aligned}$$

Let $\Theta_0(TZ) := \overline{H}_0^{-1}(\partial Df)^* \wedge (\partial Df) - \overline{H}_0^{-1}(\partial Df)^* Df \overline{H}_0^{-1} \wedge (Df)^*(\partial Df)$. By Taylor series expansion we see that

$$\begin{aligned}
i\Theta_0(TZ)(z, w) &= i\overline{H}_0^{-1}(\partial Df(z, 0) + \mathcal{O}(\|w\|))^* \wedge (\partial Df(z, 0) + \mathcal{O}(\|w\|)) \\
&\quad - i\overline{H}_0^{-1}(\partial Df(z, 0) + \mathcal{O}(\|w\|))^*(Df(z, 0) + \mathcal{O}(\|w\|)) \\
&\quad \cdot \overline{H}_0^{-1} \wedge (Df(z, 0) + \mathcal{O}(\|w\|))^*(\partial Df(z, 0) + \mathcal{O}(\|w\|)) \\
&= i\Theta_0(TZ)(z, 0) + \mathcal{O}(\|w\|)\mathcal{O}(\|z_2\|^{-\kappa} + \|z_2\|^{-2\kappa}).
\end{aligned}$$

Let $i\Theta_1(TZ)(z, w) = i\Theta(TZ)(z, w) - i\Theta_0(TZ)(z, 0)$. The form $i\Theta_0(z, 0)$ is nonpositive on a neighbourhood of $a(U \setminus S)$, conic along $a(S)$. The ‘error term’ $i\Theta_1(TZ)$ is in the worst case

$$\begin{aligned}
&\mathcal{O}(\|w\|)\mathcal{O}(\|z_2\|^{-\kappa} + \|z_2\|^{-2\kappa}) + \mathcal{O}(\|z_2\|^{2\kappa+k_1}) + \mathcal{O}(\|w\|^{l-1}\|z_2\|^{-\kappa}) \\
&+ \mathcal{O}(\|z_2\|^{\kappa+k_1}) + \mathcal{O}(\|z_2\|^{2(2\kappa+k_1)}) + \mathcal{O}(\|z_2\|^{-2\kappa})\mathcal{O}(\|w\|^{l-1})
\end{aligned}$$

and it decreases at least as $\|z_2\|^{k_1}$ on conic neighbourhoods $\|w\| \leq \|z_2\|^{4\kappa+k_1}$. \square

Proof of Corollary 3.3. The part H_0 of the Levi form of Φ near $(z, 0) \in a(S)$ that dominates in the matrix H on a cone is bounded from below by $\|z_2\|^{2k}I$. The potentially positive part of the Nakano curvature, $i\Theta_1$, is of the form

$$\mathcal{O}(\|z_2\|^{k_1})$$

on $\|w\| \leq \|z_2\|^{4\kappa+k_1}$ and can be compensated by $-\partial\bar{\partial}\Phi/2$ for $k_1 > 2k$ thus making the curvature tensor

$$(i\Theta_0(z, 0) - i\partial\bar{\partial}\Phi(z, w)) + (i\Theta_1(z, w) - i\partial\bar{\partial}\Phi(z, w))$$

strictly Nakano negative. \square

4. PROOF OF THE MAIN THEOREM

By theorem 1.1 in [Pre2] the bundle E can be endowed with a Nakano positive Hermitian metric h_0 on a conic neighbourhood of $a(U \setminus S)$ with polynomial poles on $\pi^{-1}(S)$. Let $\omega = i\partial\bar{\partial}\Phi$ be the given metric. In order to solve the $\bar{\partial}$ -equation in bidegree (p, q) we have to show that the curvature tensor

$$i\Theta(E \otimes \Lambda^s TZ) + iL\Psi = i\Theta(E) + i\Theta(\Lambda^s TZ) + iL\Psi$$

is positive (or at least nonnegative) for some strictly plurisubharmonic weight Ψ and $s = n - p$.

The Kähler metric h induced by the Kähler form ω has almost nonpositive Nakano curvature. Let h^s be the metric on $\Lambda^s TZ$ induced by h and h_1 the metric induced by the form $\omega_1 = \omega e^\Phi$. The latter has strictly Nakano negative curvature tensor

$$i\Theta(TZ)_{\omega_1} = i\Theta(TZ)_\omega - i\partial\bar{\partial}\Phi$$

by Corollary 3.3. If the original metric on TZ is represented by H then the new one is $H_1 = H e^\Phi$ and the induced metric h_1^s on $\Lambda^s TZ$ is represented by the matrix

$$H_1^s = H^s e^{s\Phi}.$$

Since the Chern curvature tensor of the Hermitian metric ω_1 on TZ is Nakano negative the induced curvature tensor on $\Lambda^s TZ_{\omega_1}$ is also Nakano negative by Lemma 3.1 and equals

$$i\Theta(\Lambda^s TZ)_{h_1^s} = i\Theta(\Lambda^s TZ)_{h^s} - i s \partial\bar{\partial}\Phi <_{\text{Nak}} 0.$$

Write $F := \Lambda^s TZ$ and let θ_F be the bilinear form on $TZ \otimes F$ associated to $i\Theta(F)$. Since the rank of the bundle F is $\binom{n}{s}$, formula (3.3) gives

$$\theta_1 = -\binom{n}{s} \text{Tr}_F(\theta_F)_{h_1^s} \otimes h_1^s + \theta_{F_{h_1^s}} >_{\text{Nak}} 0.$$

We observe that θ_1 is the curvature form associated to the Chern curvature tensor of the bundle $F \otimes (\det F^*)^{\binom{n}{s}}$ with the metric induced by h_1 . The induced metric on $\det F^*$ equals

$$(4.1) \quad h_{\det F^*} = (\det H_1)^{-\binom{n-1}{s-1}} = (\det H)^{-\binom{n-1}{s-1}} e^{-\binom{n-1}{s-1}\Phi}$$

by the identity

$$(4.2) \quad \det F = \det \Lambda^s TZ = (\det TZ)^{\binom{n-1}{s-1}}$$

and so

$$i\Theta(\det F^*)_{h_1} = i\Theta(\det F^*)_h + \binom{n-1}{s-1} i\partial\bar{\partial}\Phi.$$

Then

$$\theta_1 = \theta((\det F^*)^{\binom{n}{s}})_h \otimes h_1^s + e^{s\Phi} \theta(F)_h + \left(\binom{n}{s} \binom{n-1}{s-1} - s \right) \partial\bar{\partial}\Phi \otimes h_1^s >_{\text{Nak}} 0$$

or

$$(4.3) \quad \theta = \theta((\det F^*)^{\binom{n}{s}})_h \otimes h^s + \theta(F)_h + \left(\binom{n}{s} \binom{n-1}{s-1} - s \right) \partial\bar{\partial}\Phi \otimes h^s >_{\text{Nak}} 0.$$

To complete the proof we view this expression as part of the curvature of the metric $e^{-\Psi} h^s$ on $\Lambda^s TZ$. Let $\Phi_1 := (\binom{n}{s} \binom{n-1}{s-1} - s)\Phi$. Then the weight $e^{-\Phi_1}$ gives the last term of θ . Observe that

$$i\Theta(\det F^*)_h = i\bar{\partial}\partial \log(\det H)^{-\binom{n-1}{s-1}} = i\partial\bar{\partial} \log(\det H)^{\binom{n-1}{s-1}}.$$

Write

$$F = F \otimes (\det F^*)^{\binom{n}{s}} \otimes (\det F)^{\binom{n}{s}} = F \otimes (\det F^*)^{\binom{n}{s}} \otimes (\det TZ)^{\binom{n-1}{s-1} \binom{n}{s}}$$

by invoking (4.2) and define the Nakano positive metric on $(\det TZ)^{\binom{n-1}{s-1} \binom{n}{s}}$ in the following way. Let v_i be smooth sections of $\det TZ$, given by Proposition 2.1 in [Pre2], which are holomorphic to the degree $l_2 > 2$ with zeroes of order k_2 on $\pi^{-1}(S)$ and such that they

generate the bundle $\det TZ$ on a neighbourhood of $a(U \setminus S)$, conic along $a(S)$. Let v be a local holomorphic section, $v_i = \alpha_i v$ and define

$$\Phi_2 := \log \sum \langle v_i, v_i \rangle_H = \log \langle v, v \rangle_H + \log \sum |\alpha_i|^2.$$

Then the metric $e^{-\Psi} h^s$ for $\Psi = \Phi + \Phi_1 + \binom{n}{s} \binom{n-1}{s-1} \Phi_2$ has

$$\begin{aligned} & i \binom{n}{s} \binom{n-1}{s-1} \partial \bar{\partial} \log \det H + i \Theta(\Lambda^s TZ)_\omega + i \left(\binom{n}{s} \binom{n-1}{s-1} - s \right) \partial \bar{\partial} \Phi \\ & + i \partial \bar{\partial} \Phi + i \binom{n}{s} \binom{n-1}{s-1} \partial \bar{\partial} \log \sum |\alpha_i|^2 \end{aligned}$$

as a curvature tensor. The first three terms give a Nakano positive curvature by (4.3) and the last is also Nakano positive in a suitable conic neighbourhood because the negative part of $\partial \bar{\partial} \log \sum |\alpha_i|^2$ is of the form

$$C_1 \frac{\|w\|^{l_2}}{\|z_2\|} + C_2 \|w\|^{2l_2} + C_3 \|w\|^{l_2} + C_4 \|w\|^{l_2-1}$$

and its modulus decreases at least as $\|z_2\|^{k_1}$ on conic neighbourhoods $\|w\| \leq \|z_2\|^{k_1+2}$ (see [Pre2], p.14 for details). Because $i \partial \bar{\partial} \Phi$ is strictly plurisubharmonic with the rate of degeneracy at most $\|z_2\|^{2k}$ (independent of the shape of the cone if it is sharp enough) it compensates the negativity of $\partial \bar{\partial} \log \sum |\alpha_i|^2$ provided $k_1 > 2k$. \square

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